Calculations in étale cohomology

Sidi Mohammed Boughalem (shinokiz@gmail.com)

11. December 2019

Étale Cohomology is a cohomology theory for schemes which plays an important role in algebraic geometry and number theory. In this seminar, we will explore some concrete and useful computations of étale cohomology, highlighting its intersection with other cohomology theories.

1 Introduction

Let X be a scheme, we define the small site of X, denoted $X_{ét}$ as follows: Recall that a map is said to be *étale* if it is flat and unramified at each point of X. We define the category $X_{ét}$ by:

- Objects of $X_{ét}$ are étale X-schemes (schemes U such that $\varphi_U : U \to X$ is étale)
- Morphisms in $X_{ét}$ are étale morhisms of X-schemes $\phi : U \to V$ such that the following commutes:

$$\begin{array}{ccc} U & \xrightarrow{\phi} & V \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

A presheaf (resp. a sheaf) on $X_{ét}$ will be called an étale presheaf (resp. étale sheaf). Etale presheaves (resp. étale sheaf) on the small site of X form the category $\mathrm{PSh}_{\text{ét}}(X)$ (resp. $\mathrm{Sh}_{\text{ét}}(X)$)

Now for all $U \in X_{ét}$, $f : X \to Y$ a morphism of schemes, the functors

$$\Gamma(U, \cdot) : \mathrm{Sh}_{\text{ét}}(X) \to Ab \tag{1}$$

$$\Gamma(U, f_*(\cdot)) : \mathrm{Sh}_{\text{ét}}(X) \to Ab \tag{2}$$

are left exact. Since $\mathrm{Sh}_{\text{ét}}(X)$ is abelian and has enough injectives, one defines for $q \geq 0$ their right derived functors

$$\mathcal{F} \mapsto R^q\Gamma(U, \mathcal{F}) =: H^q_{ét}(U, \mathcal{F})$$

$$\mathcal{F} \mapsto R^q\Gamma(U, f_*\mathcal{F}) = H^q_{ét}(U \underset{X}{\times} Y, \mathcal{F})$$

Seminar „Etale cohomology “, WS 19/20, Universität Regensburg

Definition 1.1. *Let $q \geq 0$. We define the q-th étale cohomology group of an étale sheaf $\mathcal{F}$ to be the q-th derived functor*

$$\mathcal{F} \mapsto H^q_{ét}(X, \mathcal{F})$$

Similarly, we define the q-th higher direct image of an étale sheaf $\mathcal{G}$ to be q-the derived functor

$$\mathcal{G} \mapsto R^q f_* \mathcal{G}$$

associated to the presheaf

$$\mathcal{F} \mapsto H^q_{ét}(U \underset{X}{\times} Y, \mathcal{G})$$

Let X be a scheme and $\mathcal{U} : \{u_i \to U\}$ be an étale covering in $X_{ét}$. For any sheaf $\mathcal{F} \in \mathrm{Sh}_{\text{ét}}(X)$, one has the following biregular spectral sequences

$$\textit{Leray} : E_2^{p,q} = H^p_{ét}(Y, R^q f_* \mathcal{F}) \Longrightarrow H^{p+q}_{ét}(X, \mathcal{F}) \tag{3}$$

$$\textit{Čech-to-derived functor} : E_2^{p,q} = \check{\mathrm{H}}^p(Y, \mathcal{H}^q(\mathcal{G})) \Longrightarrow H^{p+q}_{ét}(X, \mathcal{G}) \tag{4}$$

Where $\mathcal{H}^q(\mathcal{G}) : U \mapsto H^q(U, \mathcal{G})$.

Definition 1.2. *Let X be a scheme. A sheaf $\mathcal{F} \in \mathrm{Sh}_{\text{ét}}(X)$ is said to be flasque if for all étale covering $\mathcal{U} : \{u_i \to U\}$ in $X_{ét}$, one has*

$$\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{F}) = 0$$

2 Properties of étale cohomology

Let X be a scheme, in this section we will prove some basic properties of étale cohomology, in particular, we will prove that for a finite morphism of schemes, the higher direct images vanish.

Proposition 2.1. *Let $f : X \to Y$ be a finite, radicial and surjective morphism of schemes. Then one has*

$$\forall \mathcal{G} \in \mathrm{Sh}_{\text{ét}}(Y) \quad H^q_{ét}(Y, \mathcal{G}) \cong H^q_{ét}(X, f^* \mathcal{G})$$

Proof. we know that, since f is finite, one has for each $y \in Y$

$$(f_* \mathcal{F})_{\overline{y}} \cong \bigoplus_{x \in X \underset{\mathscr{O}_Y}{\otimes} \overline{k(y)}} \mathcal{F}_{\overline{x}}$$

Since f is radicial, $X \underset{\mathscr{O}_Y}{\otimes} \overline{k(y)}$ consist only of one point. Let $\mathcal{F} \in \mathrm{Sh}_{\text{ét}}(X)$, then one has

$$f^*(f_* \mathcal{F})_{\overline{y}} \cong f^* \mathcal{F}_{\overline{x}} = \mathcal{F}_{f(\overline{x})}$$
$$\Rightarrow f^* f_* \mathcal{F} \cong \mathcal{F}$$

Moreover, one gets even an equivalence of categories between $\mathrm{Sh}_{\text{ét}}(X)$ and $\mathrm{Sh}_{\text{ét}}(Y)$ (see [Fu11][5.3.10]). In particular, since f_* is exact:

$$E^2_{pq} = H^{p+q}_{\acute{e}t}(Y, R^q f_* \mathcal{F}) = 0 \quad \forall q \geq 1$$

Hence,

$$H^p_{\acute{e}t}(Y, f_*\mathcal{G}) \cong H^p_{\acute{e}t}(X, \mathcal{G}) \quad \forall \mathcal{G} \in \mathrm{Sh}_{\text{ét}}(X)$$

In particular, for $\mathcal{G} = f^*\mathcal{F}$

$$H^q_{\acute{e}t}(Y, \mathcal{F}) \cong H^q_{\acute{e}t}(Y, f_*(f^*\mathcal{F})) \cong H^q_{\acute{e}t}(X, f^*\mathcal{F})$$

□

As a first consequence of this result, we can examine some useful restrictions while computing the cohomology:

Corollary 2.2. *(i) Let X be a scheme, then for any sheaf $\mathcal{F} \in \mathrm{Sh}_{\text{ét}}(X)$*

$$H^q_{\acute{e}t}(X, \mathcal{F}) \cong H^q_{\acute{e}t}(X_{red}, \mathcal{F}|_{X_{red}})$$

(ii) Let X be a scheme over a field K, and let L be a purely inseparable extension of K. Then for any sheaf $\mathcal{F} \in \mathrm{Sh}_{\text{ét}}(X)$

$$H^q_{\acute{e}t}(X, \mathcal{F}) \cong H^q_{\acute{e}t}(X \otimes_K L, \mathcal{F}|_{X \otimes_K L})$$

Proof. (i) consider the map $f : X_{red} \to X$, for simplicity, we restrict to the affine case, then we globalize it to the general one. Suppose X is affine, and consider the following commutative diagram for any open affine $U = \mathrm{Spec}(A)$

$$\begin{array}{ccc} X_{red} & \hookrightarrow & X \\ \uparrow & & \uparrow \\ f^{-1}(U) = \mathrm{Spec}\left(A/_{Nil(A)} \right) & \longrightarrow & U = \mathrm{Spec}(A) \end{array}$$

Clearly, as f is a closed immersion, it is finite and surjective, in order to show that it is radicial, consider the following diagram, where K is an algebraically closed field:

$$\begin{array}{ccc} X_{red}(K) & \longrightarrow & X(K) \\ \| & & \| \\ \mathrm{Hom}(\mathrm{Spec}(K), X_{red}) & \xrightarrow{\sim} & \mathrm{Hom}(\mathrm{Spec}(K), X) \end{array}$$

This comes form the universal property of reduction: In the affine case always, for $X = \mathrm{Spec}(A)$; a map $\alpha \in \mathrm{Hom}(A, K)$ factors uniquely into

$$A \longrightarrow A_{red} = A/_{Nil(A)} \xrightarrow{\exists !} K$$

As $Nil(A) \subseteq \ker(\alpha)$

(ii) Consider the commutative diagram

$$\begin{array}{ccc} X \otimes_K L & \xrightarrow{f} & X \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(K) \end{array}$$

Clearly, f is surjective, and finite since L/K is finite. The fact that it is radicial is a direct consequence of the following characterization: a morph$f$ is radicial if and only if it is injective, and for any $x \in X$, the residue field $k(x)$ of X at x is a purely inseparable algebraic extension of the residue field $k(f(x))$ of Y at $f(x)$. (See [Fu11][1.7.1]).

□

Proposition 2.3. *Let A be a strict Henselian local ring, $X = \mathrm{Spec}(A)$. Then for any sheaf $\mathcal{F} \in \mathrm{Sh}_{\text{ét}}(X)$*

$$H^q_{\acute{e}t}(X, \mathcal{F}) \cong 0 \quad \forall q \geq 0$$

Proof. This is due to the fact that $\Gamma(X, \cdot)$ is exact. Indeed, $\Gamma(\mathrm{Spec}(A), \mathcal{F}) \cong \mathcal{F}_s$ where s is the closed point of X, and the stalk functor is exact. □

As a final result, we show the vanishing of the higher direct images of étale sheaves, under a finite morphism.

Proposition 2.4. *Let $f : X \to Y$ be a finite morphism of schemes, and $\mathcal{F}$ an étale sheaf over X. Then one has*

- $R^q f_* \mathcal{F} = 0 \quad \forall q \geq 1$
- $H^q_{\acute{e}t}(Y, f_* \mathcal{F}) \cong H^q_{\acute{e}t}(X, \mathcal{F}) \quad \forall q \geq 0$

Proof. Since f_* is exact, $R^q f_* \mathcal{F} = 0$ and by the Leray spectral sequence (3)

$$E_2^{pq} = H^q_{\acute{e}t}(Y, f_* \mathcal{F}) \cong H^q_{\acute{e}t}(X, \mathcal{F})$$

□

3 The Étale-Zariski cohomologies

Figure 1: A Zariski-étale meme

Let X be a scheme, and X_{Zar} the site induced by the Zariski topology. In this section we want to compare the cohomology on the étale and Zarisky sites, and by 'comparing' we concretely want to define pullback and pushforward functors between the categories $X_{ét}$ and X_{Zar}. We start from the continuous map of Grothendieck topologies

$$i : X_{ét} \longrightarrow X_{Zar}$$
$$i^{-1}(U \xrightarrow{\text{étale}} X) \longmapsto U \text{ Open}$$

As a first intuition, we define pushforwards as simply *'the restriction'* from the étale to the Zariski topology, i.e

$$i_*\mathcal{F} : U \mapsto \mathcal{F}(U)$$

where U is a Zariski open and $\mathcal{F}$ an étale sheaf on X. This defines a 'Zariski' sheaf $i_*\mathcal{F}$. For the opposite direction, one might suspect that (étale) sheafification will be needed. Indeed, we define pullbacks as the étale sheaf associated to the presheaf

$$i^{\mathcal{P}}\mathcal{G} : (U \xrightarrow[ét]{f} X) \longmapsto \varinjlim_{f(U)\subseteq V} \mathcal{G}(V) = \underset{\uparrow}{\mathcal{G}}(f(V))$$

(Since étale maps are also open)

where V is an open and $\mathcal{G}$ a sheaf on X. This defines an étale sheaf $i^*\mathcal{G}$ and one gets a bijection

$$\mathrm{Hom}(\mathcal{G}, i_*\mathcal{F}) \cong \mathrm{Hom}(i^*\mathcal{G}, \mathcal{F})$$

so i_* is right adjoint to i^*. Moreover, i^* is exact as it commutes with finite limits and finite fibre products. Sheafification yields a natural map on global sections $\mathcal{F}(X) \to i^*\mathcal{F}(X)$, and by exactness of i_*, one obtains a (δ-functorial) map between cohomologies

$$H^\bullet_{Zar}(X, \mathcal{F}) \to H^\bullet_{ét}(X, i^*\mathcal{F}) \tag{5}$$

Intuitively, as the étale topology is finer than the Zariski one, one does not expect the map in (5) to be an isomorphism. However, we will see that in some particular conditions, one can make this happen.

3.1 Cohomology of étale sheaves associated to quasi-coherent modules

Let $\mathcal{F}$ be a Zariski sheaf, that is a quasi coherent $\mathscr{O}_X$-module. We define $\mathcal{F}_{ét}$ to be the étale sheaf

$$(U \xrightarrow[ét]{\pi} X) \mapsto \Gamma(U, \pi^*\mathcal{F})$$

Theorem 3.1. *Let X be a scheme, $\mathscr{M}$ a quasi coherent $\mathscr{O}_X$-module. Then one has*

$$H^q_{Zar}(X, \mathscr{M}) \cong H^q_{ét}(X, \mathscr{M}_{ét})$$

Remark 3.2. *Before proving this result, we need to make few things clear. What we implicitly do is that we use a sightly different version of $i^*\mathcal{F}$, which is an $\mathscr{O}_X$-module version of the morphism in (5). Explicitly, for the $\mathscr{O}_X$-module $\mathcal{F}$, the structure sheaf on the étale site $\mathscr{O}_{X_{\acute{e}t}}(U) := \mathscr{O}_U(U)$ yields a map of rings $\mathscr{O}_X \longrightarrow i_*\mathscr{O}_{X_{\acute{e}t}}$ on the site X_{Zar}. By adjunction, $i^*\mathscr{O}_{X_{\acute{e}t}} \longrightarrow \mathscr{O}_X$ yields a flat map on stalks*

$$\mathscr{O}_{X,x} \longrightarrow \widetilde{\mathscr{O}}_{X,\overline{x}} \cong \mathscr{O}_{X_{\acute{e}t},\overline{x}}$$

where $\widetilde{\mathscr{O}}_{X_{\acute{e}t},s}$ is the strict Henselisation of the local ring $\mathscr{O}_{X,x}$ at a geometric point $\overline{x}$ (For more details, see [Fu11][5.3]). We thus obtain that

$$\mathcal{F}_{\acute{e}t} \cong \mathscr{O}_{X_{\acute{e}t}} i^* \underset{\otimes}{\mathscr{O}_X} i^*\mathcal{F}$$

and thus, the morphism in (5) becomes

$$H^\bullet_{Zar}(X,\mathcal{F}) \to H^\bullet_{\acute{e}t}(X, i^*\mathcal{F}) \to H^\bullet_{\acute{e}t}(X,\mathcal{F}_{\acute{e}t})$$

Proof of Theorem 3.1. Let $i : X_{\acute{e}t} \to X_{Zar}$, and consider the biregular spectral sequence

$$E_2^{p,q} = H^p_{Zar}(X, R^q i_*\mathscr{M}_{\acute{e}t}) \Longrightarrow H^{p+q}_{\acute{e}t}(X, M_{\acute{e}t})$$

To show that the edge map

$$H^q_{Zar}(X,\mathscr{M}) \cong H^q_{\acute{e}t}(X,\mathscr{M}_{\acute{e}t}) \tag{6}$$

is an isomorphism, it suffices to show that $R^q i_*(\mathscr{M}_{\acute{e}t}) = 0$ for all $q \geq 1$. As $R^q i_*(\mathscr{M}_{\acute{e}t})$ is the sheaf associated to the presheaf $U \longmapsto H^q_{\acute{e}t}(U,\mathscr{M}_{\acute{e}t})$ for $U \subseteq X$ open, it is sufficient to show that $H^q_{\acute{e}t}(U,\mathscr{M}_{\acute{e}t}) = 0$ for all $q \geq 1$, affine scheme U and quasi-coherent $\mathscr{O}_U$-module $\mathscr{M}$. We proceed by induction on q.

– For $q = 1$: $\mathscr{M}_{\acute{e}t}$ is flasque over $X^{\mathrm{aff}}_{\acute{e}t}$.

  It suffices to check flasquness on finite covers. Let $\mathcal{U} := \{U_i \to U\}_{i\in I}$ be an étale covering. Checking flasqueness on $\mathcal{U}$ is equivalent to checking it on $\{\coprod U_i \to U\}_{i\in I}$, which is again equivalent to checking it on $\{V \to U\}$ where $V \to U$ is affine, faithfully flat. We make use of the following lemma:

  Lemma 3.3 ([**Fu11**] Lemma (5.7.6))**.** *Let $A \to B$ be a faithfully flat ring homomorphism, M be an A-module. Then the sequence*

$$0 \to M \xrightarrow{d_{-1}} M \otimes_A B \xrightarrow{d_0} M \otimes_A B \otimes_A B \xrightarrow{d_1} M \otimes_A B \otimes_A B \otimes_A B \xrightarrow{d_2} \dots \tag{7}$$

  is exact, where

$$d_n : x \otimes b_0 \otimes \dots \otimes b_n \longmapsto \sum_{i=1}^{n+1} (-1)^i x \otimes b_0 \otimes \dots \otimes b_{i-1} \otimes 1 \otimes \dots \otimes b_n.$$

Now, note that

$$\check{\mathrm{H}}^q(U, \mathscr{M}_{ét}) \cong \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^q(\mathcal{U}, \mathscr{M}_{ét}) = 0 \quad \forall q \geq 1 \tag{8}$$

as the Čech complexe here is nothing but (7). Thus, by the Čech-to-derived functor spectral sequence (4), one has a canonical isomorphism

$$H^1_{ét}(U, \mathscr{M}_{ét}) \cong \check{\mathrm{H}}^1(U, \mathscr{M}_{ét}) = 0$$

– For $q \geq 1$: Suppose $H^n_{ét}(U, \mathscr{M}_{ét})$ for all $1 \leq n \leq q$.

Consider once again, the biregular spectral sequence (4)

$$E_2^{p,q} = \check{\mathrm{H}}^p(U, \mathcal{H}^q(\mathscr{M}_{ét})) \Longrightarrow H^{p+q}_{ét}(U, \mathscr{M}_{ét}))$$

One has

$$\check{\mathrm{H}}^p(U, \mathcal{H}^q(\mathscr{M}_{ét})) = \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^q(\mathcal{U}, \mathscr{M}_{ét})$$

where $\mathcal{U} := \{U_i \to U\}_{i \in I}$ is an étale covering, U_i affine and I finite. Each $U_{i_1 \dots i_n} = U_{i_1} \underset{U}{\times} \dots \underset{U}{\times} U_{i_n}$ is affine, hence

$$\mathcal{H}^n(\mathscr{M}_{ét})(U_{i_1, \dots, i_n}) = H^n_{ét}(U_{i_1 \dots i_n}, \mathscr{M}_{ét})) = 0 \quad \forall n \leq q$$

Therefor $\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{H}^n(\mathscr{M}_{ét})) = 0$ and $E_2^{p,n} = 0$ for all $n \leq q$ and any p. In particular

$$E_2^{n,0} \cong H^n_{ét}(U, \mathscr{M}_{ét}))$$

and one the an exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & E_2^{0,q} & \longrightarrow E_2^{q+1,0} \longrightarrow & H^{q+1}_{ét}(U, \mathscr{M}_{ét}) & \longrightarrow & E_2^{0,q+1} & \longrightarrow \dots \\ & || & & & & || & \\ & 0 & & & & \check{\mathrm{H}}^0(U, \mathcal{H}^{q+1}(\mathscr{M}_{ét})) = 0 & \end{array}$$

Thus

$$H^{q+1}_{ét}(U, \mathscr{M}_{ét}) \cong E_2^{q+1,0} = \check{\mathrm{H}}^{q+1}(U, \mathscr{M}_{ét}) = 0 \quad \text{(by (8))}$$

□

Remark 3.4. *Note that, if the scheme X is affine, one has*

$$H^q_{ét}(X, \mathscr{M}_{ét}) \cong H^q_{Zar}(X, \mathscr{M}) = 0$$

3.2 First degree cohomology

Let X be a scheme, $Pic(X)$ is the group formed out of equivalence classes of invertible sheaves, and is canonically isomorphic to $H^1_{Zar}(X, \mathscr{O}_X^\times)$ ([Fu06], Corollary (2.3.12)). We will see in this subsection that the Zariski topology is fine enough to compute the first degree cohomology.

Theorem 3.5. *Let X be a scheme, and let $\mathscr{O}^\times_{X_{ét}}$ be the étale sheaf defined by $U \mapsto \mathscr{O}_U^\times(U)$ for all $U \in X_{ét}$. Then one has*

$$H^1_{ét}(X, \mathscr{O}^\times_{X_{ét}}) \cong H^1_{Zar}(X, \mathscr{O}^\times_X) \cong Pic(X).$$

Proof. Consider $i : X_{ét} \to X_{Zar}$. As done in the proof of theorem (3.1), one brings up the Leray biregular spectral sequence once again

$$H^p_{Zar}(X, R^q i_* \mathscr{O}^\times_{X_{ét}}) \Longrightarrow H^{p+q}_{ét}(X, \mathscr{O}^\times_{X_{ét}})$$

This induces the exact sequence for the low-degree terms

$$\begin{array}{ccccccc} 0 \longrightarrow & E_2^{1,0} & \longrightarrow & H^1_{ét}(U, \mathscr{O}^\times_{X_{ét}}) & \longrightarrow & E_2^{0,1} & \longrightarrow \dots \\ & \| & & & & \| & \\ & H^1_{Zar}(X, \mathscr{O}^\times_X) & & & & H^0_{Zar}(X, R^1 i_* \mathscr{O}^\times_{X_{ét}}) & \end{array}$$

Thus it suffices to show that $R^1 i_* \mathscr{O}^\times_{X_{ét}} = 0$, i.e. for all $x \in X$ the stalks

$$(R^1 i_* \mathscr{O}^\times_{X_{ét}})_x = 0$$

As $R^1 i_* \mathscr{O}^\times_{X_{ét}}$ is the sheaf associated to the presehaf $U \longmapsto H^1_{ét}(U, \mathscr{O}^\times_{X_{ét}})$, it is equivalent to showing that $H^1_{ét}(V, \mathscr{O}^\times_{X_{ét}}) = 0$ on stalks, i.e. for all $x \in X$,

$$\begin{aligned} \varinjlim_{x \in U} H^1_{ét}(U, \mathscr{O}^\times_{X_{ét}}) &\cong \varinjlim_{x \in U} \check{\mathrm{H}}^1_{ét}(U, \mathscr{O}^\times_{X_{ét}}) \\ &= \varinjlim_{V \subseteq U} \check{\mathrm{H}}^1(U_i \times_U V, \mathscr{O}^\times_{X_{ét}}) = 0 \end{aligned}$$

for every étale covering $\mathcal{U} : \{U_i \to U\}$ in $X_{ét}$. Therefor, we can assume that $U = \mathrm{Spec}(A)$ is affine, the U_i are affine, I is finite and $A = \mathscr{O}_{X,x}$ is a local ring. Let $V = \mathrm{Spec}(B)$, then as seen before, one gets a faitfully flat homomorphism $A \to B$, and the Čech complexe

$$0 \to B \xrightarrow{d_{-1}} B \otimes_A B \xrightarrow{d_0} B \otimes_A B \otimes_A B \xrightarrow{d_1} B \otimes_A B \otimes_A B \otimes_A B \xrightarrow{d_2} ...$$

Now let $\lfloor \gamma \rfloor$ be a class represented by the 1-cocycle $\gamma \in (B \otimes_A B)^\times$. This yields an isomorphism $\phi : B \otimes_A B \xrightarrow{\sim} B \otimes_A B$ which fulfills the co-cycle condition. By descent theory, there exists a finitely generated and flat A-module M such that

$$B \otimes_A M \cong B$$

(since the B-module B is also flat and finitely generated). Since A is local, M is a free A-module of rank 1 and $M \cong A$. By flat descent again, ϕ must come from the $B \otimes_A B$-module isomorphism

$$B \otimes_A B \otimes_A B \xrightarrow{\sim} B \otimes_A B \otimes_A B$$

and thus the associated 1-cocycle is trivial. Hence our Čech is a co-boundary and the claim follows. □

4 étale cohomology of fields.

Let k be a field, and k_s be a separable closure of k. This defines a geometric point $s : \text{Spec}(k_s) \to \text{Spec}(k)$. We set $G_k := Gal\left({}^{k_s}/_{k}\right)$. Given a sheaf $\mathcal{F}$ on $\text{Spec}(k)_{ét}$, one has

$$\mathcal{F}_s = \varinjlim_{\substack{k_s|k'|k \\ k'|k \text{ finite, galois}}} \mathcal{F}(\text{Spec}(k') \quad \text{(See [Tam94][p116-118])}$$

We will show in this section that the étale cohomology over $Spec(k)$ is nothing else but the classical Galois cohomology. This provides an alternative way to compute group cohomology of discrete G_k-modules. But first, we recall some facts about Galois theory:

4.1 preliminaries on Galois theory

Let X be a scheme. We define a (left) action of G_k on $X(k_s) := \text{Hom}_k(\text{Spec}(k_s), X)$ by $(\varphi, f) \mapsto f \circ \varphi$, for every $\varphi \in G_k$ and every $f \in X(k_s)$ (On the topological spaces elements of G_k just send a point to itself). We recall the well known following facts:

- G_k is a profinite group, i.e a projective limit of finite groups

$$G_k = \varprojlim_{\substack{k_s|k'|k \\ k'|k \text{ finite, galois}}} Gal\left({}^{k'}/_{k}\right)$$

- For an open subgroup $G_i \leq G_k$ one has $X(k_s)^{G_i} = X(k_s^{G_i})$. Morover one has a decomposition

$$X(k_s) = \coprod_{G_i \leq G_k} X(k_s^{G_i})$$

  and G_k acts continuously on $X(k_s)$.

- A G_k-set M is called discrete if for all $m \in M$, $\text{Stab}_{G_k}(m)$ is open.

- For an étale $\text{Spec}(k)$-scehem Y, one has the decomposition

$$Y = \coprod_{i \in I} \text{Spec}(k_i) \quad \text{where } {}^{k_i}/_{k} \text{ are finite, separable field extensions.} \tag{9}$$

4.2 Galois-étale cohomology

Let k be a field, k_s its separable closure and s be the corresponding geometric point. We are now ready to prove the main theorem of this section, relating étale cohomology over Spec(k) and Galois cohomology.

Theorem 4.1. *Under the notations above, the functor $\mathcal{F} \mapsto \mathcal{F}_s$ induces an equivalence of categories*

$$\mathrm{Spec}(k)_{\acute{e}t} \longrightarrow {}_{G_k}Mod$$

Moreover, one has for all q

$$H^q_{\acute{e}t}(\mathrm{Spec}(k), \mathcal{F}) \cong H^q(Gal\left({}^{k_i'}/_{k}\right), \mathcal{F}_s)$$

In order to prove this result we will proceed as follows:

Step 1: One has an equivalence of categories given by:

$$\begin{aligned} \mathrm{Spec}(k)_{\acute{e}t} &\longrightarrow G_k\text{-Sets} \\ X &\overset{\varphi}{\longmapsto} X(k_s) \\ \coprod_{i\in I} \mathrm{Spec}(k_s^{G_i}) &\overset{\phi}{\longleftarrow\!\!\shortmid} M \end{aligned}$$

Proof. We first show that our maps are well defined. It is clear that φ is. For a discrete G_k set M, then one has a G_k-orbit decomposition

$$M \cong \coprod_{i\in I} M_i$$

Since G_k is profinite, the open sets $G_i = Stab_{G_k}(m_i) \leq G_k$ are closed and of finite index, for all $m_i \in M$. By (infinite) Galois theory, one gets that $k_s^{G_i}/k$ are finite and separable field extensions, hence the morphisms

$$\phi_i : \mathrm{Spec}(k) \to \mathrm{Spec}(k_s^{G_i})$$

are unramified. They are clearly flat as well, hence étale (over k). As Spec$(k)_{\acute{e}t}$ has arbitrary coproducts, $\phi = \bigcup \phi_i$ is étale (over k). Moreover one can easily show that ϕ does not depend on the choice of the points in the orbits: indeed, a choice of a different point in M_i is the same as substituting the field extensions $k_s^{G_i}$ with one of its G_k-conjugates, thus one is done by passing to spectra.
Now we need to show that φ and ϕ are quasi inverses of each others. Let X be an étale k-scheme. By (9)

$$X = \coprod_{i\in I} \mathrm{Spec}(k_i) \quad \text{where } {}^{k_i}/_{k} \text{ are finite, separable field extensions.}$$

Now, fixing a separable closure k_s on has

$$\phi(X) = X(k_s) = (\coprod_{i\in I} \mathrm{Spec}(k_i))(k_s) = \coprod_{i\in I} \mathrm{Spec}(k_i)(k_s)$$

Now, for $i \in I$

$$\mathrm{Spec}(k_i)(k_s) \cong \mathrm{Hom}_k(k_i, k_s)$$

and G_k acts transitively on $\mathrm{Hom}_k(k_i, k_s)$ by permutating the $k_i \hookrightarrow k_s$. Thus, $\varphi \circ \phi(X)$ corresponds to the étale k-scheme

$$\coprod_{i \in I} \mathrm{Spec}(k_s^{G_i})$$

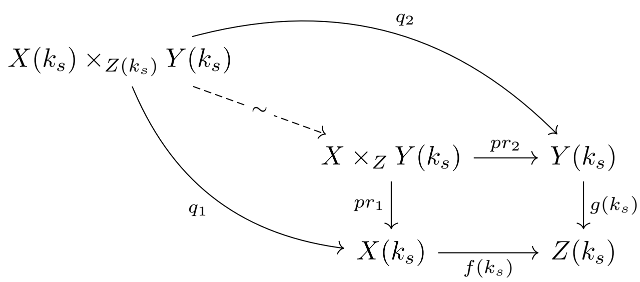
where $G_i = Stab_{G_k}(f_i)$, f_i a chosen point in $\mathrm{Spec}(k_i)(k_s)$. But clearly, G_i is also the stabiliser of the action on G_k i.e. $G_i = Gal\left({}^{k_s}\!/_{k_i}\right)$. Finally, $k_s^{G_i} = k_s^{Gal(k_s/k_i)} = k_i$ and one has $\varphi \circ \phi(X) = X$.
The opposite direction is clear since ϕ commutes with coproducts. □

Remark 4.2. *One actually has to show slightly more here, namely that the equivalence above induces an isomorphism of sites*

$$\mathrm{Spec}(k)_{\acute{e}t} \cong \mathcal{T}_{G_k}$$

Where $\mathcal{T}_{G_k}$ is the canonical site on the category G_k-Sets. This boils down essentially to showing that

- *The following is a pulback diagram in G_k-Sets: Given maps $(X \xrightarrow{f} Z)$ and $(Y \xrightarrow{g} Z)$ in $\mathrm{Spec}(k)_{\acute{e}t}$*



- *For all covering $\{U_i \to U\}$ one has a covering $\{\phi(U_i) \to \phi(U)\}$ in $\mathrm{Spec}(k)_{\acute{e}t}$. This comes from essential surjectivity and full faithfullnes of ϕ.*

Step 2: one has the following equivalence of categories between discrete G_k-modules and sheaves of abelian groups on the site $\mathcal{T}_{G_k}$, given by:

$$Sh_{Ab}(\mathcal{T}_{G_k}) \xrightarrow{\sim} {}_{G_k}\mathrm{Mod}^{\mathrm{disc}} \tag{10}$$

$$\mathcal{F} \overset{\varphi}{\longmapsto} \varinjlim_{\substack{G_i \trianglelefteq G_k \\ \text{open}}} \mathcal{F}(G_k/G_i)$$

$$\mathscr{H}om_{G_k}(\,\cdot\,, M) \overset{\phi}{\longleftarrow} M$$

Proof. This comes actually from a more general result:

Proposition 4.3 ([Tam94] Prop (1.3.3.1) or [Jan15] Theorem (8.7))**.** *Let G be a profinite group. Then the following functors yield an equivalence of categories*

$$Sh(\mathcal{T}_{G_k}) \xrightarrow{\sim} G_k\text{-}Sets$$
$$\mathcal{F} \overset{\varphi}{\longmapsto} \varinjlim_{\substack{G_i \trianglelefteq G_k \\ open}} \mathcal{F}(G_k/G_i)$$
$$\mathrm{Hom}_{G_k}(\,\cdot\,, M) \overset{\phi}{\longleftarrow} M$$

□

In particular, by Step 1, we get an equivalence

$$Sh^{ab}_{\acute{e}t}(\mathrm{Spec}(k)) \cong {}_{G_k}\mathrm{Mod}^{\mathrm{disc}}$$

And by the first results of section 4.1

$$\mathrm{Hom}_{G_k}\left({}^{G_k}\!/_{G_i}, X(k_s)\right) \cong X(k_s)^{G_i} \cong X(k_s^{G_i}) \cong \mathrm{Hom}_{X_{\acute{e}t}}(\mathrm{Spec}(k_s^{G_i}, X)$$

This induces the equivalence of categories

$$Sh^{ab}_{\acute{e}t}(\mathrm{Spec}(k)) \xrightarrow{\sim} {}_{G_k}\mathrm{Mod}^{\mathrm{disc}}$$
$$\mathcal{F} \longmapsto \mathcal{F}_s = \varprojlim_{\substack{k_s|k'|k \\ k'|k \text{ finite, galois}}} \mathcal{F}(\mathrm{Spec}(k')$$
$$M^{(\cdot)} \longleftarrow M$$

where $s : \mathrm{Spec}(k_s) \to \mathrm{Spec}(k)$.

Step 3: For every (abelian) sheaf $\mathcal{F}$ on $\mathrm{Spec}(k)_{\acute{e}t}$ and every $q \geq 0$ we have a (δ -functorial) isomorphism

$$H^q_{\acute{e}t}(\mathrm{Spec}(k), \mathcal{F}) \cong H^q(G_k, \varprojlim_{k'} \mathcal{F}(\mathrm{Spec}(k'))$$

Proof. Let $\mathcal{F}$ be an (abelian) sheaf on the site $\mathrm{Spec}(k)_{\acute{e}t}$, corresponding to the G_k-module $M \cong \mathcal{F}_s$ via (10). Then one has

$$\begin{aligned}\Gamma(\mathrm{Spec}(k), \mathrm{Hom}_{G_k}(\,\cdot\,, M)) &\cong \mathrm{Hom}_{Sh^{ab}_{\acute{e}t}(\mathrm{Spec}(k))}(h_{\mathrm{Spec}(k)}, \mathrm{Hom}_{G_k}(\,\cdot\,, M)) \\ &\cong \mathrm{Hom}_{G_k}(\{*\}, M) \cong M^{G_k}\end{aligned}$$

where $h_{Spec(k)} = \mathrm{Hom}_{\mathrm{Spec}(k)_{\acute{e}t}}(\,\cdot\,, \mathrm{Spec}(k))$. Thus we get

$$\begin{aligned} H^q_{\acute{e}t}(\mathrm{Spec}(k), \mathcal{F}) &\cong R^q M^{G_k} = H^q(\{*\}, \mathcal{F}') \quad \text{(where } \mathcal{F}' \text{ corresponds to } \mathcal{F} \text{ under (10))} \\ &\cong H^q(\{*\}, \mathrm{Hom}_{G_k}(\,\cdot\,, M)) \\ &\cong H^q(G_k, M) \cong H^q\left(G_k, \varinjlim_{\substack{G_i \trianglelefteq G_k \\ \text{open}}} \mathcal{F}'(G_k/G_i)\right) \\ &\cong H^q\left(G_k, \varprojlim_{\substack{k_s | k' | k \\ k'|k \text{ finite, galois}}} \mathcal{F}(\mathrm{Spec}(k'))\right) \\ &\cong H^q(G_k, \mathcal{F}_s). \end{aligned}$$

□

Finally, we end this section by presenting a nice application of what we have done so far, which is considered to be one of the main results in classical Galois Cohomology.

Corollary 4.4 (Hilbert 90)**.** *Let k be a field, then one has*

$$H^1(Gal({}^{k_s}/_{k}), k_s^\times) = 0$$

Proof. By theorem (4.1), for every (abelian) sheaf $\mathcal{F}$ on $\mathrm{Spec}(k)_{\acute{e}t}$

$$H^q_{\acute{e}t}(\mathrm{Spec}(k), \mathcal{F}) \cong H^q(Gal\left({}^{k'_i}/_{k}\right), \mathcal{F}_s)$$

Now let X be an étale scheme consider the presheaf $\mathcal{G} := \mathscr{O}_X^\times : X \mapsto \mathscr{O}_X^\times(X)$. Then

$$\mathcal{G} \cong \mathrm{Hom}_{X_{\acute{e}t}}(\,\cdot\,, X \underset{\mathrm{Spec}(\mathbb{Z})}{\times} \mathrm{Spec}\,\mathbb{Z}[T, T^{-1}])$$

Hence $\mathcal{G} \in Sh_{\acute{e}t}(X)^{ab}$ and for $X = \mathrm{Spec}(k)$ we get

$$\begin{aligned} \mathcal{G}_s = \varprojlim_{\substack{k_s | k' | k \\ k'|k \text{ finite, galois}}} \mathcal{G}(\mathrm{Spec}(k')) &= \varprojlim_{\substack{k_s | k' | k \\ k'|k \text{ finite, galois}}} \mathscr{O}^\times_{\mathrm{Spec}(k')}(\mathrm{Spec}(k')) \\ &= \varprojlim_{\substack{k_s | k' | k \\ k'|k \text{ finite, galois}}} k' = k_s^\times \end{aligned}$$

thus

$$\begin{aligned} H^1(G_k, k_s^\times) \cong H^1(G_k, (\mathscr{O}^\times_{\mathrm{Spec}(k)})_s) &\cong H^1_{\acute{e}t}\left(\mathrm{Spec}(k), \mathscr{O}^\times_{\mathrm{Spec}(k)_{\acute{e}t}}\right) \\ &\cong H^1_{Zar}\left(\mathrm{Spec}(k), \mathscr{O}^\times_{\mathrm{Spec}(k)}\right) \quad \text{by Theorem (3.1)} \\ &\cong Pic(\mathrm{Spec}(k)) = 0 \end{aligned}$$

The last assertion comes from the following: Let $\mathcal{F}$ be an invertible sheaf on $X = \operatorname{Spec}(k)$, and let $\mathcal{G}$ be the $\mathscr{O}_X$-module such that $\mathcal{F} \otimes_{\mathscr{O}_X} \mathcal{G} \cong \mathscr{O}_X$. Then the last assertion is equivalent to proving that $\mathcal{F} \cong \mathscr{O}_X$. We do that on the stalks (well, stalk here since we have only one point)

$$\mathcal{F}_x \otimes_{\mathscr{O}_{X,x}} \mathcal{G}_x \cong (\mathcal{F} \otimes_{\mathscr{O}_X} \mathcal{G})_x \cong \mathscr{O}_{X,x} \cong k$$

Hence $\mathcal{F}_x \cong k$ and we are done. □

References

[Fu06] L. Fu. *Algebraic Geometry*. Mathematics series for graduate students. Tsinghua University Press, 2006.

[Fu11] L. Fu. *Etale Cohomology Theory*. Nankai tracts in mathematics. World Scientific, 2011.

[Jan15] U. Jannsen. Étale cohomology/étale kohomologie. `http://www.mathematik.uni-regensburg.de/Jannsen/Etale-gesamt-eng.pdf`, Lecture notes SS 2015.

[Sta20] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2020.

[Tam94] G. Tamme. *Introduction to Étale Cohomology*. Universitext (Berlin. Print). Springer-Verlag, 1994.